NIU Physics PhD Candidacy Exam – Standard Formula Sheet

Pauli Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Definite Integrals

For $n =$ non-negative integer,

$$\begin{aligned}
\int_0^\infty x^n e^{-x}\, dx &= n! \\
\int_0^\infty x^{2n} e^{-\beta x^2}\, dx &= \frac{(2n)!\,\sqrt{\pi}}{n!\,(2\sqrt{\beta})^{2n+1}} \\
\int_0^\infty x^{2n+1} e^{-\beta x^2}\, dx &= \frac{n!}{2\,\beta^{n+1}}
\end{aligned}$$

where $0! = 1$ and $n! = 1 \cdot 2 \ldots (n-1) \cdot n$.

Stirling's Approximation:

$$\begin{aligned}
n! &= \sqrt{2\pi}\, n^{n+1/2} \exp\left(-n + \frac{1}{12n} + \mathcal{O}(1/n^2)\right) \\
\ln(n!) &= n \ln(n) - n \qquad\qquad (\text{for } n \gg 1)
\end{aligned}$$

Legendre Polynomials:

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = (3x^2 - 1)/2, \qquad P_3(x) = (5x^3 - 3x)/2,$$

$$P_4(x) = (35x^4 - 30x^2 + 3)/8, \qquad P_5(x) = (63x^5 - 70x^3 + 15x)/8$$

Spherical Harmonics:

$$\begin{aligned}
\ell = 0: &\quad Y_0^0(\theta,\phi) = \sqrt{\frac{1}{4\pi}} \\
\ell = 1: &\quad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}} e^{\pm i\phi} \sin\theta \\
\ell = 2: &\quad Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \qquad Y_2^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}} e^{\pm i\phi} \sin\theta\cos\theta \\
&\quad Y_2^{\pm 2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\phi} \sin^2\theta \\
\text{For all } \ell: &\quad Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)
\end{aligned}$$

Numerical Constants:

$\hbar = 1.05 \times 10^{-27}\,\text{erg sec} = 1.05 \times 10^{-34}\,\text{J sec}$

$a_0 = 0.529 \times 10^{-8}\,\text{cm} = 0.529 \times 10^{-10}\,\text{m}$

$\hbar c = 1.97 \times 10^{-7}\,\text{eV m}$

$hc = 1.24 \times 10^{-6}\,\text{eV m}$

$e = 4.80 \times 10^{-10}\,\text{esu} = 1.60 \times 10^{-19}\,\text{C}$

$c = 3.00 \times 10^{10}\,\text{cm/sec} = 3.00 \times 10^{8}\,\text{m/sec}$

$\epsilon_0 = 8.85 \times 10^{-12}\,\text{C}^2/\text{N m}^2$

$\mu_0 = 4\pi \times 10^{-7}\,\text{N/A}^2$

$m_e = 9.11 \times 10^{-28}\,\text{g} = 9.11 \times 10^{-31}\,\text{kg} = 0.511\,\text{MeV}/c^2$

$m_p = 1.67262 \times 10^{-24}\,\text{g} = 1.67262 \times 10^{-27}\,\text{kg} = 938.272\,\text{MeV}/c^2$

$m_n = 1.67492 \times 10^{-24}\,\text{g} = 1.67492 \times 10^{-27}\,\text{kg} = 939.565\,\text{MeV}/c^2$

$N_0 = 6.02 \times 10^{23}\,\text{particles/mole}$

$k_B = 1.38 \times 10^{-23}\,\text{J K}^{-1} = 1.38 \times 10^{-16}\,\text{erg K}^{-1} = 8.62 \times 10^{-5}\,\text{eV K}^{-1}$

Spherical Coordinates (r, θ, ϕ)

Relations to rectangular (Cartesian) coordinates and unit vectors:

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \qquad \begin{aligned} \hat{x} &= \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi \\ \hat{y} &= \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi \\ \hat{z} &= \hat{r}\cos\theta - \hat{\theta}\sin\theta \end{aligned}$$

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi &= \tan^{-1}(y/x) \end{aligned} \qquad \begin{aligned} \hat{r} &= \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta \\ \hat{\theta} &= \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta \\ \hat{\phi} &= -\hat{x}\sin\phi + \hat{y}\cos\phi \end{aligned}$$

Line element: $d\vec{\ell} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$

Volume element: $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\vec{\nabla} f = \dfrac{\partial f}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\hat{\theta} + \dfrac{1}{r\,\sin\theta}\dfrac{\partial f}{\partial \phi}\hat{\phi}$

Divergence: $\vec{\nabla}\cdot\vec{v} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 v_r) + \dfrac{1}{r\,\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta\,v_\theta) + \dfrac{1}{r\,\sin\theta}\dfrac{\partial v_\phi}{\partial\phi}$

Curl:
$$\vec{\nabla}\times\vec{v} = \frac{1}{r\,\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r\,v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

Laplacian: $\nabla^2 f = \dfrac{1}{r}\dfrac{\partial^2}{\partial r^2}(rf) + \dfrac{1}{r^2\,\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\,\dfrac{\partial f}{\partial\theta}\right) + \dfrac{1}{r^2\,\sin^2\theta}\dfrac{\partial^2 f}{\partial\phi^2}$

Cylindrical Coordinates (r, ϕ, z)

Relations to rectangular (Cartesian) coordinates and unit vectors:

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned} \qquad \begin{aligned} \hat{x} &= \hat{r}\cos\phi - \hat{\phi}\sin\phi \\ \hat{y} &= \hat{r}\sin\phi + \hat{\phi}\cos\phi \\ \hat{z} &= \hat{z} \end{aligned}$$

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \phi &= \tan^{-1}(y/x) \\ z &= z \end{aligned} \qquad \begin{aligned} \hat{r} &= \hat{x}\cos\phi + \hat{y}\sin\phi \\ \hat{\phi} &= -\hat{x}\sin\phi + \hat{y}\cos\phi \\ \hat{z} &= \hat{z} \end{aligned}$$

Line element: $d\vec{\ell} = \hat{r}\,dr + \hat{\phi}\,r\,d\phi + \hat{z}\,dz$

Volume element: $d\tau = r\,dr\,d\phi\,dz$

Gradient: $\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$

Divergence: $\vec{\nabla}\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\vec{\nabla}\times\vec{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}\right]\hat{z}$

Laplacian: $\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$

Vector Formulae

In the following formulae, $\vec{A}$ and $\vec{B}$ are vector functions and ψ is a scalar function.

$$\begin{aligned}
\vec{\nabla}\times(\vec{\nabla}\psi) &= 0 \\
\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) &= 0 \\
\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) &= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} \\
\vec{\nabla}(\vec{A}\cdot\vec{B}) &= (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) \\
\vec{\nabla}\cdot(\vec{A}\times\vec{B}) &= \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) \\
\vec{\nabla}\times(\vec{A}\times\vec{B}) &= \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A}) + (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} \\
\vec{\nabla}\cdot(\psi\vec{A}) &= \vec{A}\cdot\vec{\nabla}\psi + \psi\vec{\nabla}\cdot\vec{A} \\
\vec{\nabla}\times(\psi\vec{A}) &= \psi\vec{\nabla}\times\vec{A} - \vec{A}\times\vec{\nabla}\psi
\end{aligned}$$